

FRITZ-JOHN AND KUHN-TUCKER CONDITIONS IN NON-LINEAR PROGRAMMING

K. JEYALAKSHMI

Department of Mathematics, SRM University, Chennai, Tamil Nadu, India

ABSTRACT

In this paper we give necessary optimality conditions of Fritz-John and Kuhn-Tucker (KKT) conditions for non-linear infinite dimensional programming problem with operatorial constraints. We use an alternative theorem. Some of the known results in finite dimensional case have been extended to infinite dimensional case with suitable conditions.

KEYWORDS: Non-Linear Programming, Fritz-John Conditions, Karush-Kuhn-Tucker Conditions, Alternative Theorem

1. INTRODUCTION

The role of optimality criteria in mathematical programming is important from both theoretical and computational point of view. We consider here the general mathematical programming problem (P) given by

(P) Min $f(x)$ subject to

$G(x) \leq 0, x \in A$, where A is a subset of X .

Here $f: X \rightarrow \mathbb{R}$ is a function and $G: X \rightarrow Y$ is an operator which may be differentiable, convex, non-smooth or non-convex and Y is a partially ordered vector space ordered by a closed convex cone with interior points.

Many authors have investigated the optimality conditions for problem (P). It was Kuhn and Tucker who first established the necessary condition for (P) for differentiable functions with

$Y = \mathbb{R}$. Kannappan [10] established the Fritz-John and KKT conditions for convex objective and constraint functions and give some duality results also. Programming problems involving generalized convexity like invex, d-invex, preinvex, convexlike, quasi-convex etc have been investigated by many authors like Hirriatr-Urruty[6], Borewein, Craven and Mond, Clarke to name a few. Refer [1-3] and [6-11].

The purpose of this paper is to study programming problem involving a class of non-convex functions which are characterized by their directional derivatives. They are called directionally differentiable functions. Following A.D.Ioffe and T.M.Tihomirov[8] we call them locally convex functions.

We do not assume the differentiability of the functions involved. We derive Fritz-John and KKT type optimality conditions. In section 2, we give some definitions and a theorem of the alternative. In section 3, we establish Fritz-John necessary condition and assuming Slater's constraint qualification we prove the KKT conditions in section 4.

2. PRELIMINARIES

Throughout the paper, we let X and Y be locally convex topological Hausdorff spaces over \mathbb{R} . Let C be a closed convex cone with interior points so that (Y, C) is a partially ordered topological space and a neighborhood system $\{V\}$ of the origin such that

$V = (V+C) \cap (V-C)$. Such a cone is known as normal cone. The dual cone C^* is given by

$$C^* = \{y^* \in Y^* / \langle y^*, y \rangle \geq 0, \forall y \in C\}.$$

We follow the convention: for $y_1, y_2 \in Y$,

$$y_1 \leq y_2 \implies y_2 - y_1 \in C$$

$$y_1 < y_2 \implies y_2 - y_1 \in C^0$$

Definition 2.1: Let X be a locally convex space. A function $f: X \rightarrow \mathbb{R}$ is said to be a locally convex at $x_0 \in X$ if (i) the one sided directional derivative of f at x_0 in the direction of $x \in X$ given by

$$f'(x_0; x) = \lim_{\lambda \rightarrow 0^+} \frac{f(x_0 + \lambda x) - f(x_0)}{\lambda} \text{ exists for all } x \in X;$$

(ii) $x \rightarrow f'(x_0; x)$ is convex and continuous.

Remark 2.1: Any continuous convex function is locally convex. However, there are locally convex functions which are not convex.

Ex: $h: \mathbb{R} \rightarrow \mathbb{R}$ defined by $h(x) = \log(1 + |x|)$. Then f is locally convex function. However it is not convex.

Refer [4,8].

Definition 2.2: Let $G: X \rightarrow Y$ be an operator. If the limit

$$\lim_{\lambda \rightarrow 0^+} \frac{G(x_0 + \lambda d) - G(x_0)}{\lambda}$$

exists, then it is called the one-sided directional derivative of G at x_0 in the direction d , and it is denoted by $G'(x_0; d)$.

It is clear that $G'(x_0; \cdot)$ is a positively homogenous operator.

An operator $G: X \rightarrow Y$ is said to be locally convex at $x_0 \in X$, if (i) $G'(x_0; d)$ exists for all

$$x \in X,$$

(ii) $x \rightarrow G'(x_0; x)$ is continuous and convex.

Definition 2.3: Let $f: X \rightarrow \mathbb{R}$ be a locally convex function and $x_0 \in X$. The subdifferential of f at x_0 denoted $\partial f(x_0)$ is defined by

$$\partial f(x_0) = \{x^* \in X^* / \langle x^*, x \rangle \leq f'(x_0; x) \forall x \in X\}$$

It is clear that $\partial f(x_0) = \partial f'(x_0; 0)$.

The elements of $\partial f(x_0)$ are called the sub gradients of f at x_0 and it is a w^* -compact subset of X^* . [4].

Definition 2.4: Let $G: X \rightarrow Y$ be a locally convex operator. A continuous linear operator

$T: X \rightarrow Y$ is said to be a sub gradient of G at x_0 if $T(x) \leq G(x_0, x)$ for all $x \in X$. The set of all subgradients of G at x_0 is called the sub differential of G at x_0 and is denoted by $\partial G(x_0)$. G is said to be sub differentiable at x_0 if $\partial G(x_0)$ is non-empty.

Definition 2.5: A locally convex operator $G: X \rightarrow Y$ is said to be regularly sub differentiable at $x_0 \in X$, if G is sub differentiable at x_0 and satisfies

$$\partial(y^* \circ G)(x_0) = y^* \circ \partial G(x_0) = \{y^* \circ T / T \in \partial G(x_0), \text{ for all } y^* \in C^*\}$$

Moreau-Rockafellar Theorem

Let f_1, f_2, \dots, f_n be convex functions on a locally convex space X . Then

$\partial f_1(x) + \partial f_2(x) + \dots + \partial f_n(x) \subseteq \partial(f_1 + \partial f_2 + \dots + f_n)$ for every $x \in X$. If at a point $x_0 \in X$, all the functionals f_1, f_2, \dots, f_n except possibly one are continuous, then

$$\partial f_1(x) + \partial f_2(x) + \dots + \partial f_n(x) = \partial(f_1 + \partial f_2 + \dots + f_n).$$

3. NECESSARY OPTIMALITY CONDITIONS OF FRITZ-JOHN TYPE

Let f be a continuous locally convex functional defined on a locally convex space X , and let

G be a continuous locally convex space Y . It is assumed that Y is ordered by a closed convex cone C with non-empty interior. We shall also assume that G is regularly sub differentiable on A , a convex subset of X . We consider the following programming problem:

(P) Minimize $f(x)$ subject to

$$G(x) \leq 0, x \in A.$$

In this section, we prove a necessary Fritz-John optimality conditions for a vector to be optimal for (P).

First, we prove the following Lemma.

Lemma 3.1: Let A, X, Y and C be as defined above. Let G be operator. Then, either (a) or (b) of the following holds:

- There is $x_0 \in A$ such that $G(x_0) < 0$;
- There is $y^* \in C^*$ such that $y^* \neq 0$ and $\langle G(x), y^* \rangle \geq 0$, for all $x \in A$.

Proof: Clearly (a) and (b) exclude each other. Now suppose (a) does not hold.

$$\text{Then } G(A) \cap \text{int}(-C) = \varnothing.$$

Claim: $\text{conv}(G(A)) \cap \text{int}(-C) = \varnothing$, where $\text{conv}(G(A))$ denotes the convex hull of $G(A)$.

Suppose for some $0 < \alpha < 1$, and x_0 and $y_0 \in A$,

$$\alpha G(x_0) + (1 - \alpha) G(y_0) \in \text{int}(-C).$$

Then $K = [\alpha G(x_0) + (1 - \alpha) G(y_0), -(\alpha G(x_0) + (1 - \alpha) G(y_0))]$ is a neighbourhood of zero in Y [25]. As '+' and '-' are continuous, there exists a neighbourhood $n(\alpha)$ around α and neighborhoods $n(G(x_0)), n(G(y_0))$ of $G(x_0)$ and $G(y_0)$ respectively, such that for all $u \in n(G(x_0)), v \in n(G(y_0))$ and $\mu \in n(\alpha), \mu u + (1 - \mu)v \in K$.

In particular, for some $\mu < \alpha$, $\mu G(x_0) + (1 - \mu)G(y_0) \in K$.

(i.e.) $\alpha G(x_0) + (1 - \alpha)G(y_0) \leq \mu G(x_0) + (1 - \mu)G(y_0)$

(i.e.) $(\alpha - \mu)G(x_0) \leq (\alpha - \mu)G(y_0)$

(i.e.) $G(y_0) \leq G(x_0)$.

It follows that $G(y_0) \leq \alpha G(x_0) + (1 - \alpha)G(y_0) < 0$, which is a contradiction to the assumption.

Hence $\text{conv}(G(A)) \cap \text{int}(-C) = \emptyset$.

So, by Hahn-Banach separation theorem, there exists a

$y^* \in C^*$, $y^* \neq 0$ such that $\inf y^*(\text{conv}(G(A))) \geq \sup y^*(-C) = 0$.

Since C is a convex cone, $\sup y^*(-C) = 0$

Hence $y^* \in C^*$ and $\langle G(x), y^* \rangle \geq 0$, for all $x \in A$.

Hence the Lemma.

Theorem 3.1: If x_0 is an optimal solution of the problem (P), then there exists $\lambda_0 \geq 0$, $y_0^* \in C^*$, not both

zero, such that $0 \in \partial f(x_0) + y_0^* \partial G(x_0) + N(x_0/A)$ (1)

and

$\langle G(x_0), y_0^* \rangle = 0$. (2)

Proof: Let $H: A - x_0 \rightarrow \mathbb{R} \times Y$ be defined by $H(x) = (f'(x_0; x), G'(x_0; x) + G(x_0))$

It is clear that H is convex.

We claim that there is no $\bar{x} \in A - x_0$ such that $H(\bar{x}) < 0$.

Suppose there is $\bar{x} \in A - x_0$ such that $H(\bar{x}) < 0$.

Then $f'(x_0; \bar{x}) < 0$ and $G'(x_0; \bar{x}) + G(x_0) < 0$

Since x_0 is a solution of (P), $f(x_0) \leq f(x_0 + \lambda \bar{x})$ for all sufficiently small λ .

Hence $f'(x_0 + \lambda \bar{x}) \geq 0$ which is a contradiction.

Thus there is no $\bar{x} \in A - x_0$ such that $H(\bar{x}) < 0$. From Lemma I-1, there exists $\lambda_0 \geq 0$ and

$y_0^* \in C^*$, not both zero, such that

$\lambda_0 f'(x_0; x) + \langle G'(x_0; x), y_0^* \rangle + \langle G(x_0), y_0^* \rangle \geq 0$ for all $x \in A - x_0$ (3)

Setting $x = 0$ in (3), we get $\langle G(x_0), y_0^* \rangle \geq 0$ (4)

Since $G(x_0) \leq 0$ and $y_0^* \in C^*$, we also have $\langle G(x_0), y_0^* \rangle \leq 0$. (5)

From (4) and (5), we have

$\langle G(x_0), y_0^* \rangle = 0$. (6)

So, (3) reduces to $\lambda_0 f'(x_0; x) + \langle G'(x_0; x), y_0^* \rangle \geq 0$, for all $x \in A - x_0$

That is,

$$\lambda_0 f'(x_0; x) + \langle G'(x_0; x), y_0^* \rangle \geq \lambda_0 f'(x_0; 0) + \langle G'(x_0; 0), y_0^* \rangle$$

$$\text{for all } x \in A - x_0. \quad (7)$$

$$\lambda_0 f'(x_0; x) + \langle G'(x_0; x), y_0^* \rangle + \delta(x/A - x_0) \geq$$

$$\lambda_0 f'(x_0; 0) + \langle G'(x_0; 0), y_0^* \rangle + \delta(0/A - x_0). \quad (8)$$

This implies that the function

$$\lambda_0 f'(x_0; \cdot) + \langle G'(x_0; \cdot), y_0^* \rangle + \delta(\cdot/A - x_0) \text{ attains its minimum at zero.}$$

$$\text{Hence, } 0 \in \partial (\lambda_0 f'(x_0; \cdot) + y_0^* \circ G'(x_0; \cdot) + \delta(\cdot/A - x_0)). \quad (9)$$

By Moreau-Rockafellar theorem

$$0 \in \lambda_0 \partial f(x_0) + \partial (y_0^* \circ G)(x_0) + N(x_0/A). \quad (10)$$

Since G is regularly subdifferentiable on A , we have $\partial (y_0^* \circ G) = y_0^* \circ \partial G(x_0)$

Therefore (10) reduces to

$$0 \in \lambda_0 \partial f(x_0) + y_0^* \circ \partial G(x_0) + N(x_0/A)$$

This proves the theorem.

4. NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS OF KUHN-TUCKER TYPE

In theorem 3.1, if λ_0 is zero, then the objective function f has no role to play in the necessary conditions. To avoid this undesirable situation, we have to guarantee that λ_0 is strictly positive.

To achieve this, we assume the following generalized constraint qualification of Slater's type [1].

Definition 4.1: We say that a constraint qualification of *Slater's* type is satisfied by a programming problem of type (P), if there exists $x' \in A$ such that $G(x') < 0$. That is, $-G(x')$ is an interior point of the cone C .

Note 4.1: Throughout the subsequent sections we assume the Slater's constraint qualification.

Theorem 4.1: Let X be a locally convex space and f a continuous locally convex functional defined on a convex set $A \subseteq X$ and let Y be an ordered locally convex space with positive cone C with non- empty interior. Let $x_0 \in A$. Let $G: X \rightarrow Y$ be continuous and locally convex on A . Let G also be regularly sub differentiable on A . x_0 is an optimal solution of (P), if and only if there exist

$y_0^* \in C^*$ such that

$$0 \in \partial f(x_0) + y_0^* \circ \partial G(x_0) + N(x_0/A), \quad (11)$$

$$\text{and } \langle G(x_0), y_0^* \rangle = 0. \quad (12)$$

Proof

Necessity: By Theorem I-1, there exist $\lambda_0 \geq 0$, $y_0^* \in C^*$, not both zero such that

$$\lambda_0 f'(x_0; x) + \langle G'(x_0; x), y_0^* \rangle \geq 0, \text{ for all } x \in A - x_0. \quad (13)$$

$$\langle G(x_0), y_0^* \rangle = 0. \quad (14)$$

Suppose $\lambda_0 = 0$. Then $\langle G'(x_0; x), y_0^* \rangle \geq 0$, for all $x \in A - x_0$ (15)

Since Slater's constraint qualification is satisfied, there is $x' \in A$ such that $G(x') < 0$.

There is \bar{x} such that $\bar{x} + x_0 = x'$ and $G(\bar{x} + \lambda x_0) < 0$, for some $\lambda > 0$.

From (14), it follows that $G'(x_0, \bar{x}) < 0$. (16)

Thus

$$\langle G'(x_0; \bar{x}), y_0^* \rangle = 0. \quad (16)$$

Since $G'(x_0; \bar{x}) < 0$, there exists a symmetric neighborhood U of zero in Y such that

$$-G'(x_0; \bar{x}) + U \subseteq C.$$

Hence for every $u \in U$, we would have

$$\langle u, y_0^* \rangle \leq \langle G'(x_0; \bar{x}), y_0^* \rangle = 0$$

which implies that $y_0^* = 0$ which is a contradiction to $(\lambda_0, y_0^*) \neq (0, 0)$

This proves the necessary part.

Sufficiency

Suppose x_0 is not optimal for (P).

(11) implies by definition,

$$f'(x_0; x) + \langle G'(x_0; x), y_0^* \rangle \geq 0, \text{ for all } x \in A - x_0. \quad (17)$$

Since we have assumed that x_0 is not optimal for (P), there exists $\bar{x} \in A$ such that

$$G(\bar{x}) \leq 0 \text{ and } f(\bar{x}) - f(x_0) < 0. \quad (18)$$

Let $d = \bar{x} - x_0 \in A - x_0$.

$$\text{Then (18) reduces to } f(x_0 + d) - f(x_0) < 0. \quad (19)$$

From (12) and (18), it follows that $\langle G'(x_0; \bar{x}), y_0^* \rangle \leq 0$

From (17), we see that $f'(x_0; d) \geq 0$ which contradicts our assumption.

This proves the sufficiency.

Remark 4.1: If $Y = \mathbb{R}^m$ and the functions involved are convex then Theorem 4.2 reduces to the following theorem.

Theorem 4.2: In problem (P), suppose f, g_i are continuous and convex functionals, $i = 1, 2, \dots, m$, A is a convex subset of X and for some $x' \in A$, $g_i(x') < 0$ for $i = 1, 2, \dots, m$. \bar{x} is an optimal solution for problem (P) if and only if there exist non-negative constants, $\lambda_i, i = 1, 2, \dots, m$ such that

$$\lambda_i g_i(\bar{x}) = 0, i = 1, 2, \dots, m \text{ and}$$

$$0 \in \partial f(\bar{x}) + \sum \lambda_i \partial g_i(\bar{x}) + N(\bar{x} / A)$$

Remark 4.2: The necessity part of the above theorem 4. 2 was proved for a convex function by Schechter [11] using the theory of Dublovitski – Milytin .

CONCLUSIONS

It is a known fact that the literature of mathematical programming is crowded with the necessary and sufficient conditions of Frit-John and Kuhn-Tucker type for point to be an optimal solution of a non-differentiable convex programming problem. In this paper, we have proved these conditions for a non-convex programming problem in a general locally convex space. Our method of proof is simpler than the classical method of convex and applicable to a larger class of functions. For instance, consider a class of locally Lipchitian functions. Clarke [2] has developed a nice theory of generalized gradients leading to sub differentiability which reduces to classical sub differentiability in convex case. Clarke's generalized gradients were extended to general locally convex spaces by Gwinner [5]. By suitable additional assumptions, the necessary conditions can be extended to these functions.

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